Confining properties of the classical
$SU(3)$ Yang - Mills theory

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Abstract

The spherically and cylindrically symmetric solutions of the $SU(3)$ Yang - Mills theory are obtained. The corresponding gauge potential has the confining properties. It is supposed that: a) the spherically symmetric solution is a field distribution of the classical "quark" and in this sense it is similar to the Coulomb potential; b) the cylindrically symmetric solution describes a classical field "string" (flux tube) between two "quarks". It is noticed that these solutions are typically for the classical $SU(3)$ Yang - Mills theory in contradiction to monopole that is an exceptional solution. This allows to conclude that the confining properties of the classical $SU(3)$ Yang - Mills theory are general properties of this theory.

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I. INTRODUCTION

In quantum chromodynamic some field configurations can be interesting for phenomenological bag theory being used for an explanation of quarks confinement. These configurations have to satisfy to the $SU(3)$ Yang - Mills equations and to confine inside itself the quark. For this purpose the $SU(2)$ magnetic bags have been constructed in Ref's [1], [2] and [3] where it has been shown that they can contain the fermions in some compact surface. In Ref. [4] it has been also shown that the quantum particles with gauge charge can be confined only inside the domain of the $SU(2)$ Yang - Mills field configuration with infinite energy. Recently the spherically symmetric solutions for the $SU(2)$ Yang - Mills equations, which are analogous to the black hole gravitational configurations, are founded in Ref's [5], [6].

In this article I want to show that the classical solutions of the $SU(3)$ Yang - Mills theory can have the confining properties and some solutions can be considered as a single quark (color charge) and a string (flux tube) between two quarks. To do this we examine the spherically and cylindrically symmetric cases.

II. SPHERICALLY SYMMETRIC CASE

The ansatz for the $SU(3)$ gauge field we take as in [7]:

$$A^a = \frac{2\varphi(r)}{Ir^2}\left(\lambda^2 x - \lambda^5 y + \lambda^7 z\right) + \frac{1}{2}\lambda^a\left(\lambda^a_{ij} + \lambda^a_{ji}\right)\frac{x^i x^j}{r^2}w(r), \tag{1a}$$

$$A^a_i = \left(\lambda^a_{ij} - \lambda^a_{ji}\right)\frac{x^j}{Ir^2}\left(f(r) - 1\right) + \lambda^a_{jk}\left(\epsilon_{ilj}x^k + \epsilon_{ilk}x^j\right)\frac{x^l}{r^3}v(r), \tag{1b}$$

here λ^a are the Gell - Mann matrixes; $a = 1, 2, \ldots, 8$ is color index; Latin indexes $i, j, k, l = 1, 2, 3$ are the space indexes; $I^2 = -1$; r, θ, φ are the spherically coordinate system. Substituting Eq's (1) in the Yang - Mills equations:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{a\mu}{}_\nu\right) + f^{abc}F^{b\mu}{}_\nu A^c_\mu = 0, \tag{2}$$

we receive the following $SU(3)$ equations system for $f(r), v(r), w(r)$ and $\varphi(r)$ functions:

$$r^2 f'' = f^3 - f + 7fv^2 + 2vw\varphi - f\left(w^2 + \varphi^2\right), \tag{3a}$$

$$r^2 v'' = v^3 - v + 7vf^2 + 2fw\varphi - v\left(w^2 + \varphi^2\right), \tag{3b}$$

$$r^2 w'' = 6w\left(f^2 + v^2\right) - 12fv\varphi, \tag{3c}$$

$$r^2 \varphi'' = 2\varphi\left(f^2 + v^2\right) - 4fvw. \tag{3d}$$

This set of equations is very difficult even for numerical investigations. We will investigate a more simpler case when only two functions are nonzero. It is easy to see that there can be only three cases. The first case is well-known monopole case by $(f, w = 0)$ or $(v, w = 0)$. Let us examine the two another interesting cases.

A. The $SU(3)$ bag

In this case we have $w = \varphi = 0$ condition. Then, the set (3) has the following view:

$$r^2 f'' = f^3 - f + 7fv^2, \tag{4a}$$

$$r^2 v'' = v^3 - v + 7vf^2. \tag{4b}$$

We seek the solutions which are regular at origin and infinity. This means that the solution at origin $r = 0$ can be expanded into a series:

$$f = 1 + f_2 \frac{r^2}{2!} + \dots, \tag{5a}$$

$$v = v_3 \frac{r^3}{3!} + \dots. \tag{5b}$$

After substitution of Eq's (5) into Eq's (4) we receive that f_2 and v_3 coefficients are arbitrary. On the infinity we seek solutions as a following series:

$$f = 1 + \frac{f_{-1}}{r} + \dots, \tag{6a}$$

$$v = \frac{v_{-2}}{r^2} + \dots. \tag{6b}$$

Analogously, the substitution of Eq's (6) into Eq's(4) shows that f_{-1} and v_{-2} coefficients are also arbitrary.

The numerical integration of the set (4) with initial date (5) shows that the solution is singular $(f \to \infty, v \to \infty)$ by some $r = r_1$. In this case the approximate analytical solution near the singularity has the following form:

$$v \approx f \approx \frac{A}{r_1 - r},$$ (7)

here $r < r_1$ and $A > 0$ is some constant.

The numerical integration of the set (4) with initial data (6) shows that here, too, the point $r = r_2$ exists in which the solution is singular: $(f \to \infty, v \to \infty)$. In this case the approximate analytical solution near the r_2 point has the following view:

$$v \approx f \approx \frac{B}{r - r_2},$$ (8)

here $r > r_2$ and $B > 0$ is also some constant.

Selecting coefficients f_2 and v_3 (or f_{-1} and v_{-2}), one can achieve the execution of $r_1 = r_2$ condition. The according solution of the set (4) (function $v(r)$) is presented on Fig.1 by $f_2 = 0.2; v_3 = 0.6; f_{-1} = 2.07; v_{-2} \approx 3.8; r_1 = r_2 \approx 2.75; f(0) = f(\infty) = 1; v(0) = v(\infty) = 0$. The graph of $f(r)$ function is practically the same. This solution is a bag with singular walls by $r = r_1 = r_2$.

The similar solution for the $SU(2)$ gauge field is received in [6].

B. The $SU(3)$ bunker

Here we examine $f = \varphi = 0$ case. The case $v = \varphi = 0$ is analogous. Now the input equations have the following form:

$$r^2 v'' = v^3 - v - v w^2,$$ (9a)

$$r^2 w'' = 6 w v^2.$$ (9b)

We seek the regular solution near $r = 0$ point. The Eq's (9) demand that v and w functions have the following view at origin $r = 0$:

$$v = 1 + v_2 \frac{r^2}{2!} + \ldots, \tag{10a}$$

$$w = w_3 \frac{r^3}{3!} + \ldots . \tag{10b}$$

The numerical integration of Eq's (9) is displayed on Fig.2,3. The asymptotical behaviour of received solution $(r \to \infty)$ is as follows:

$$v \approx a \sin \left(x^{1+\alpha} + \phi_0 \right), \tag{11a}$$

$$w \approx \pm \left[(1+\alpha)x^{1+\alpha} + \frac{\alpha}{4} \frac{\cos \left(2x^{1+\alpha} + 2\phi_0 \right)}{x^{1+\alpha}} \right], \tag{11b}$$

$$3a^2 = \alpha(\alpha + 1). \tag{11c}$$

here $x = r/r_0$ is dimensionless radius; r_0, ϕ_0 are some constants; $\alpha \approx 0.37$. For our potential A_μ^a we have the following nonzero color "magnetic" and "electric" fields:

$$H_\varphi^a \propto v', \tag{12a}$$

$$H_\theta^a \propto v', \tag{12b}$$

$$E_r^a \propto \frac{rw' - w}{r^2}, \tag{12c}$$

$$E_\varphi^a \propto \frac{vw}{r}, \tag{12d}$$

$$E_\theta^a \propto \frac{vw}{r}, \tag{12e}$$

$$H_r^a \propto \frac{v^2 - 1}{r^2}, \tag{12f}$$

here for Eq's (12a), (12b) and (12c) the color index $a = 1, 3, 4, 6, 8$ and for Eq's (12d), (12e) and (12f) $a = 2, 5, 7$. Analyzing the asymptotical behaviour of the $H_\varphi^a, H_\theta^a, H_r^a$ and E_φ^a, E_θ^a fields we see that they are the strongly oscillating fields. It is interesting that the radial components of the "magnetic" and "electric" fields drop to zero variously at infinity:

$$H_r^a \approx \frac{1}{r^2}, \tag{13a}$$

$$E_r^a \approx \frac{1}{r^{1-\alpha}}. \tag{13b}$$

Among all the (12) fields only the radial components of "electric" fields are nonoscillating. From Eq's (11) we see that our solution has the oscillating part (11a) and confining potential (11b). It is necessary to mark that obtained solution is solution with arbitrary initial

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condition, whereas the monopole solution is a special case of the initial condition. This allows to make a conclusion that the general spherically symmetric solution in the classical Yang - Mills theory has the confining properties. Although, at the same time there are the oscillating potential, "electric" and "magnetic" fields. The expression for an energy density has the following view:

$$\epsilon \propto 4\frac{v'^2}{r^2} + \frac{2}{3}\left(\frac{w'}{r} - \frac{w}{r^2}\right) + 4\frac{v^2 w^2}{r^4} + \frac{2}{r^2}\left(v^2 - 1\right). \tag{14}$$

This function is displayed on the Fig.3.

We note the asymptotical form of the energy density followed from (12) condition:

$$\epsilon \approx \frac{2}{3}\frac{\alpha(1 + \alpha)^2(3\alpha + 2)}{x^{2-2\alpha}}. \tag{15}$$

In the first approximation ϵ is nonoscillating on the infinity. The asymptotical form of ϵ leads to that the energy of such solution is infinity.

It is very interesting what happens after quantization. It is very possible that such procedure smooths these strongly oscillating $SU(3)$ gauge field. In this case we will have only nonoscillating confining potential (11b) and moreover, also nonoscillating energy density defined according to (15).

What is the physical meaning of this solution? It is possible that it is analogous to the Coulomb potential in electrostatics. But an electron can exist in empty space while a quark is not observable in a free state. Therefore, the obtained solution can describe the color charge - "quark". The quotation marks indicate that we examine the simplified Eq's (4) instead of complete Eq's (3). We emphasize that these two solutions have the fundamental difference among themselves. The electron has a singularity at origin by $r = 0$, but the "quark" at infinity by $r = \infty$. In the second case such field configuration has the confining properties. It should be also noted that this solution has the asymptotical freedom property since at origin $r = 0$ the gauge potential $A_\mu^a \to 0$.

III. THE GAUGE "STRING"

Let us write down the potential as follows:

$$A_t^2 = f(\rho), \tag{16a}$$

$$A_z^5 = v(\rho), \tag{16b}$$

$$A_\varphi^7 = \rho w(\rho), \tag{16c}$$

here we introduce the cylindrical coordinate system z, ρ, φ. The color index $a = 2, 5, 7$ corresponds to the chosen $SU(2) \subset SU(3)$ inclusion. The Yang - Mills equations for potential (16) are:

$$f'' + \frac{f'}{\rho} = f\left(v^2 + w^2\right), \tag{17a}$$

$$v'' + \frac{v'}{\rho} = v\left(-f^2 + w^2\right), \tag{17b}$$

$$w'' + \frac{w'}{\rho} - \frac{w}{\rho^2} = w\left(-f^2 + v^2\right), \tag{17c}$$

Let us examine the simplest case $w = 0$. In this case equation set is:

$$f'' + \frac{f'}{\rho} = fv^2, \tag{18a}$$

$$v'' + \frac{v'}{\rho} = -vf^2. \tag{18b}$$

At origin $\rho = 0$ the solution has the following form:

$$f = f_0 + f_2\frac{\rho^2}{2} + \dots, \tag{19a}$$

$$v = v_0 + v_2\frac{\rho^2}{2} + \dots. \tag{19b}$$

Substituting Eq's(19) into (18) system we find that:

$$f_2 = \frac{1}{2}f_0v_0^2, \tag{20a}$$

$$v_2 = -\frac{1}{2}v_0f_0^2. \tag{20b}$$

The numerical integration is shown on Fig.2,4. It can be also shown that asymptotical behaviour of f, v function and energy density ϵ is:

$$f \approx 2 \left[x + \frac{\cos\left(2x^2 + 2\phi_1\right)}{16x^3} \right], \tag{21a}$$

$$v \approx \sqrt{2} \frac{\sin\left(x^2 + \phi_1\right)}{x}, \tag{21b}$$

$$\epsilon \propto f'^2 + v'^2 + f^2 v^2 \approx const, \tag{21c}$$

here $x = \rho/\rho_0$ is dimensionless radius; ρ_0, ϕ_1 are some constants. Again we have the confining potential $A_t^2 = f(r)$ and strongly oscillating potential $A_z^5 = v(r)$. Thus, this solution is a hollow or hump (in accordance with relation between v_0 and f_0 value) on the background of the constant energy density. On account of its cylindrical symmetry we can call this solution as a "string". The quotation marks indicate that this is the string from energetically point of view, not from potential A_μ^a and field $F_{\mu\nu}^a$ point of view. It can be expected that after quantization the oscillating functions will vanish and only confining potential and constant energy density will stay.

It can be supposed that the obtained solution describes the classical gauge field between two "quarks". This must be verified using the lattice calculations (nonperturbative quantization).

IV. DISCUSSION

Thus, we receive the classical solutions of the $SU(3)$ Yang - Mills theory. These solutions show us that the confinement is a general property in the classical $SU(3)$ Yang - Mills theory. They are the typical solutions for the (3) set since $f_2, v_2, v_3, f_{-1}, v_{-2}$ constants are arbitrary. In contradiction of this the monopole solution is the exceptional solution. It has the finite energy whereas the typical solutions (obtained here) have infinite potential at infinity, infinite energy and consequently the confining properties. The confining solutions of this kind have the strong oscillating component of gauge potential, "electric" and "magnetic" fields. But it can be expected that these are only the classical properties vanishing in quantum theory.

It can be supposed that the received here solutions have the physical significance: The spherically symmetric case describes either a bag confining the quantum test particle or a

classical single "quark"; the cylindrically symmetric case describes a classical field distribution between two "quarks".

In second and third cases, the situation greatly differs from that what happens with electron. An isolated electron exists in nature and it generates the electric field decreases at infinity. An isolated quark does not exist in nature. It is possible that it forms the confining field distribution. The two interacted electrons generate electric field appearing as a superposition of electric field from two electrons. The two quarks generate the string. It can be supposed that the gauge "string" obtained above is a classical model of such field distribution. It appears as a string on the background of the field with constant energy density. Just the same it should be remarked that here we have also the strong oscillating fields. These fields maybe will be excluded by quantization.

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Fig.1. $v(r)$ function for the $SU(3)$ bag.

Fig.2. The confining potentials. 1 is the $w(r)$ function for the $SU(3)$ bunker, 2 is the $f(\rho)$ function for the $SU(3)$ "string".

Fig.3. The $SU(3)$ bunker. 1 is the oscillating potential $v(r)$ and 2 is the energy density $\epsilon(r)$.

Fig.4. The $SU(3)$ "string". 1 is the oscillating potential $v(\rho)$ and 2 is the energy density $\epsilon(\rho)$.

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